Algebraic Semigroups are Strongly π-regular

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Abstract

Let S be an algebraic semigroup (not necessarily linear) defined over a field F. We show that there exists a positive integer n such that x^n belongs to a subgroup of $S(F)$ for any $x \in S(F)$. In particular, the semigroup $S(F)$ is strongly π-regular.

1 Introduction

A fundamental result of Putcha (see [2, Thm. 3.18]) states that any *linear* algebraic semigroup S over an algebraically closed field k is strongly π-regular. The proof follows from the corresponding result for $M_n(k)$ (essentially the Fitting decomposition), combined with the fact that S is isomorphic to a closed subsemigroup of $M_n(k)$, for some $n > 0$. At the other extreme it is easy to see that any *complete* algebraic semigroup S is strongly π-regular. It is therefore natural to ask whether *any* algebraic semigroup is strongly π-regular. The purpose of this note is to provide an affirmative answer to this question, over an arbitrary field F.

2 The Main Results

Theorem 2.1. *Let S be an algebraic semigroup defined over a subfield F of k. Then the semigroup $S(F)$ is strongly π-regular, that is for any $x \in S(F)$, there exists a positive integer n and an idempotent $e \in S(F)$ such that x^n belongs to the unit group of $eS(F)e$.*

Proof. We use the terminology and results of [3, Chap. 11] for algebraic varieties defined over a field.

To show the assertion, we may replace S with any closed subsemigroup defined over F and containing some power of x. Denote by $\langle x \rangle$ the smallest closed subsemigroup of S containing x, that is, the closure of the subset $\{x^m, m > 0\}$; then $\langle x \rangle$ is defined over F by [3, Lem. 11.2.4]. The subsemigroups $\langle x^n \rangle$, $n > 0$, form a family of closed subsets of S, and satisfy $\langle x^{mn} \rangle \subseteq \langle x^m \rangle \cap \langle x^n \rangle$. Thus, there exists a smallest such semigroup, say $\langle x^{n_0} \rangle$. Replacing x with x^{n_0}, we may assume that $S = \langle x \rangle = \langle x^n \rangle$ for all $n > 0$.

Lemma 2.2. *With the above notation and assumptions, xS is dense in S. Moreover, S is irreducible.*

Proof. Since $S = \langle x^2 \rangle$, the subset $\{x^n, n \geq 2\}$ is dense in S. Hence xS is dense in S by an easy observation (Lemma 2.4) that we will use repeatedly.

Let $S_1, \ldots, S_r$ be the irreducible components of S. Then each xS_i is contained in some component S_j. Since xS is dense in S, we see that xS_i is dense in S_j. In particular, j is unique and the map $\sigma : i \mapsto j$ is a permutation. By induction, $x^n S_i$ is dense in $S_{\sigma^n(i)}$ for all n and i; thus $x^n S_i$ is dense in S_i for some n and all i. Choose i such that $x^n \in S_i$. Then it follows that $x^{mn} \in S_i$ for all m. Thus, $\langle x^m \rangle \subseteq S_i$, and $S = S_i$ is irreducible. $\square$

Lemma 2.3. *Let S be an irreducible algebraic semigroup and let $x \in S$. Assume that $S = \langle x \rangle$ (in particular, S is commutative), xS is dense in S, and S is irreducible. Then S is a monoid and x is invertible.*

Proof. For $y \in S$, consider the decreasing sequence

$$\cdots \subseteq \overline{y^{n+1}S} \subseteq \overline{y^n S} \subseteq \cdots \subseteq \overline{yS} \subseteq S$$

of closed, irreducible ideals of S. We claim that

$$\overline{y^d S} = \overline{y^{d+1} S} = \cdots ,$$

where $d := \dim(S) + 1$. Indeed, there exists $n \leq d$ such that $\overline{y^{n+1}S} = \overline{y^n S}$, that is, $y^{n+1}S$ is dense in $\overline{y^n S}$. Multipliying by y^{m-n} and using Lemma 2.4, it follows that $y^{m+1}S$ is dense in $\overline{y^m S}$ for all $m \geq n$ and hence for $m \geq d$. This proves the claim.

We may thus set

$$I_y := \overline{y^d S} = \overline{y^{d+1} S} = \cdots$$

Then we have for all $y, z \in S$,

$$\overline{y^d I_z} = I_{yz} \subseteq I_z,$$

since $y^d(z^d S) = (yz)^d S \subseteq z^d S$. Also, note that $I_x = S$, and $I_e = eS$ for any idempotent e of S. By [1, Sec. 2.3], S has a smallest idempotent e_S, and $e_S S$ is the smallest ideal of S. In particular, $e_S S \subseteq I_y$ for all y. Define

$$\mathcal{I} = \{I \subseteq S \mid I = I_y \text{ for some } y \in S\}.$$

This is a set of closed, irreducible ideals, partially ordered by inclusion, with smallest element $e_S S$ and largest element S. If $S = e_S S$, then S is a group and we are done. Otherwise, we may choose $I \in \mathcal{I}$ which covers $e_S S$ (since $\mathcal{I} \setminus \{e_S S\}$ has minimal elements under inclusion). Consider

$$T = \{y \in S \mid yI \text{ is dense in } I\}.$$

If $y, z \in T$ then $\overline{yzI} = \overline{y\overline{zI}} = I$ and hence T is a subsemigroup of S. Also, note that $T \cap e_S S = \emptyset$, since $e_S z I \subseteq e_S S$ is not dense in I for any $z \in S$. Furthermore $x \in T$. (Indeed, xS is dense in S and hence $xy^d S$ is dense in $\overline{y^d S}$ for all $y \in S$. Thus, $x\,\overline{y^d S}$ is dense in $\overline{y^d S}$; in particular, xI is dense in I).

We now claim that

$$T = \{y \in S \mid y^d I \not\subseteq e_S S\}.$$

Indeed, if $y \in T$ then $y^d I$ is dense in I and hence not contained in $e_S S$. Conversely, assume that $y^d I \not\subseteq e_S S$ and let $z \in S$ such that $I = I_z$. Since $\overline{y^d I} = \overline{y^d I_z} = I_{yz} \in \mathfrak{I}$ and $\overline{y^d I} \subseteq I$, it follows that $\overline{y^d I} = I$ as I covers $e_S S$.

By that claim, we have

$$S \setminus T = \{y \in S \mid y^d I \subseteq e_S S\} = \{y \in S \mid e_S y^d z = y^d z \text{ for all } z \in I\}.$$

Hence $S \setminus T$ is closed in S. Thus, T is an open subsemigroup of S; in particular, T is irreducible. Moreover, since $x \in T$ and xS is dense in S, it follows that xT is dense in T; also note that $\{x^n, n > 0\}$ is dense in T.

Let $e_T \in T$ be the minimal idempotent, then $e_T \notin e_S S$ and hence the closed ideal $e_T S$ contains strictly $e_S S$. Since both are irreducible, we have $\dim(e_T T) = \dim(e_T S) > \dim(e_S S)$. Now the proof is completed by induction on $\kappa(S) := \dim(S) - \dim(e_S S)$. Indeed, if $\kappa(S) = 0$, then $S = e_S S$ is a group. In the general case, we have $\kappa(T) < \kappa(S)$. By the induction assumption, T is a monoid and x is invertible in T. As T is dense in S, the neutral element of T is also neutral for S, and hence x is invertible in S. ☐

By Lemmas 2.2 and 2.3, there exists n such that $\langle x^n \rangle$ is a monoid defined over F, and x^n is invertible in that monoid. To complete the proof of Theorem 2.1, it suffices to show that the neutral element e of $\langle x^n \rangle$ is defined over F. For this, consider the morphism

$$\phi : S \times S \longrightarrow S, \quad (y, z) \longmapsto x^n y z.$$

Then ϕ is the composition of the multiplication

$$\mu : S \times S \longrightarrow S, \quad (y, z) \longmapsto yz$$

and of the left multiplication by x^n; the latter is an automorphism of S, defined over F. So ϕ is defined over F as well, and the fiber $Z := \phi^{-1}(x^n)$ is isomorphic to $\mu^{-1}(e)$, hence to the unit group of S. In particular, Z is smooth. Moreover, Z contains (e, e), and the tangent map

$$d\phi_{(e,e)} : T_{(e,e)}(S \times S) \longrightarrow T_{x^n} S$$

is surjective, since

$$d\mu_{(e,e)} : T_{(e,e)}(S \times S) = T_e S \times T_e S \longrightarrow T_e S$$

is just the addition. So Z is defined over F by [3, Cor. 11.2.14]. But Z is sent to the point e by μ. Since that morphism is defined over F, so is e. ☐

Lemma 2.4. *Let X be a topological space, and $f : X \to X$ a continuous map. If $Y \subseteq X$ is a dense subset then $f(Y) \subseteq \overline{f(X)}$ is a dense subset.*

Proof. Let $U \subseteq \overline{f(X)}$ be a nonempty open subset. Then $f^{-1}(U) \subseteq X$ is open, and nonempty since $f(X)$ is dense in $\overline{f(X)}$. Hence $Y \cap f^{-1}(U) \neq \emptyset$. If $y \in Y \cap f^{-1}(U)$ then $f(y) \in f(Y) \cap U$. Hence $f(Y) \cap U \neq \emptyset$. ☐

Remark 2.5. Given $x \in S$, there exists a *unique* idempotent $e = e(x) \in S$ such that x^n belongs to the unit group of eSe for some $n > 0$. Indeed, we then have $x^n S x^n \subseteq eSe$; moreover, since there exists $y \in eSe$ such that $x^n y = y x^n = e$, we also have $eSe = x^n y S y x^n e \subseteq x^n S x^n$. Thus, $x^n S x^n = eSe$. It follows that $x^{mn} S x^{mn}$ is a monoid with neutral element e for any $m > 0$, which yields the desired uniqueness.

In particular, if $x \in S(F)$ then the above idempotent $e(x)$ is an F-point of the closed subsemigroup $\langle x \rangle$. We now give some details on the structure of the latter semigroup. For x, e, n as above, we have $x^n = ex^n = (ex)^n$, and $y(ex)^n = e$ for some $y \in H_e$ (the unit group of $e\langle x \rangle$). But then $ex \in H_e$ since $(y(ex)^{n-1})(ex) = e$. Thus, $ex^m = (ex)^m \in H_e$ for all $m > 0$. But if $m \geq n$ then $x^m = ex^m$. Thus, if $x \notin H_e$ then there exists an unique $r > 0$ such that $x^r \notin H_e$ and $x^m \in H_e$ for any $m > r$. In particular, $x^r \in e\langle x \rangle$ for all $m \geq r$. Thus *we can write*

$$\langle x \rangle = e\langle x \rangle \sqcup \{x, x^2, ..., x^s\}$$

for some $s < r$. Notice also that *these* x^i*'s, with* $i \leq s$*, are all distinct* (if $x^i = x^j$ with $1 \leq i < j \leq s$, then $x^{i+s+1-j} = x^{s+1} \in e\langle x \rangle$, a contradiction). Moreover, a similar decomposition holds for the semigroup of F-rational points.

The set $\{ex^m, m > 0\}$ is dense in $e\langle x \rangle$ by Lemma 2.4. But $ex^m = (ex)^m$, and $ex \in H_e$. So $e\langle x \rangle$ *is a unit-dense algebraic monoid.* Furthermore, if $\langle x^{m_0} \rangle$ is the smallest subsemigroup of $\langle x \rangle$ of the form $\langle x^m \rangle$, for some $m > 0$, then $\langle x^{m_0} \rangle$ *is the neutral component of* $e\langle x \rangle$ (the unique irreducible component containing e). Indeed, $\langle x^{m_0} \rangle$ is irreducible by Lemma 2.2, and $y^{m_0} \in \langle x^{m_0} \rangle$ for any $y \in \langle x \rangle$ in view of Lemma 2.4. Thus, the unit group of $\langle x^{m_0} \rangle$ has finite index in the unit group of $\langle x \rangle$, and hence in that of $e\langle x \rangle$.

Finally, we show that Theorem 2.1 is self-improving by obtaining the following stronger statement:

Corollary 2.6. *Let S be an algebraic semigroup. Then there exists $n > 0$ (depending only on S) such that $x^n \in H_{e(x)}$ for all $x \in S$, where $e : x \mapsto e(x)$ denotes the above map. Moreover, there exists a decomposition of S into finitely many disjoint locally closed subsets U_j such that the restriction of e to each U_j is a morphism.*

Proof. We first show that for any irreducible subvariety X of S, there exists a dense open subset U of X and a positive integer $n = n(U)$ such that $x^n \in H_{e(x)}$ for all $x \in U$, and $e|_U$ is a morphism. We will consider points of S with values in the function field $k(X)$, and view them as rational maps $X - \to S$; the semigroup law on the set $S(k(X))$ of these points is given by pointwise multiplication of rational maps. In particular, the inclusion of X in S yields a point $\xi \in S(k(X))$ (the image of the generic point of X). By Theorem 2.1, there exist a positive integer n and points $e, y \in S(k(X))$ such that $e^2 = e$, $\xi^n e = e \xi^n = \xi^n$, $ye = ey = y$ and $\xi^n y = y \xi^n = e$. Let U be an open subset of X on which both rational maps $e, y : X - \to S$ are defined. Then the above relations are equalities of morphisms $U \to S$, where ξ is the inclusion. This yields the desired statements.

Next, start with an irreducible component X_0 of S and let U_0 be an open subset of X_0 such that $e|_{U_0}$ is a morphism. Now let X_1 be an irreducible component of $X_0 \setminus U_0$ and iterate this construction. This yields disjoint locally closed subsets $U_0, \ldots, U_n$ such that $e|_{U_j}$ is a morphism, and $X \setminus (U_0 \cup \cdots \cup U_j)$ is closed for all j. Hence $U_0 \cup \cdots \cup U_j = X$ for $j \gg 0$. $\square$

References

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